

The Classification of n -Lie Algebras *

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Abstract: This paper proves the isomorphic criterion theorem for $(n+2)$ -dimensional n -Lie algebras, and gives a complete classification of $(n+1)$ -dimensional n -Lie algebras and $(n+2)$ -dimensional n -Lie algebras over an algebraically closed field of characteristic zero.

Key words: n -Lie algebra, classification, multiplication table.

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1. Introduction

In 1985, Filippov [3] introduced the concept of n -Lie algebras and classified the $(n+1)$ -dimensional n -Lie algebras over an algebraically closed field of characteristic zero. The structure of n -Lie algebras is very different from that of Lie algebras due to the n -ary multilinear operations involved. The $n = 3$ case, i.e. 3-ary multilinear operation, first appeared in Nambu's work [1] in the description of simultaneous classical dynamics of three particles. In that work, Nambu extended the Poisson bracket and arrived at the generalized Hamiltonian equation involving a 3-ary multilinear bracket $\{ , , \}$. Takhtajan [2] investigated the geometrical and algebraic aspects of the generalized Nambu mechanics, and established the connection between the Nambu mechanics and Filippov's theory of n -Lie algebras [3].

The development of n -Lie algebras has opened a new chapter in the study of Lie theory, attracting much attention in different research areas due to their close connections with dynamics, geometries as well as string and membrane theories. For example, Bagger and Lambert [4] proposed a field theory model for multiple M2-branes based on the metric n -Lie algebras, and the authors in [5] found new 3-Lie algebras and their applications in membranes. More applications of the n -Lie algebras can be found in [6, 7, 8, 9, 10, 11, 12, 13].

It is known that up to isomorphisms there is a unique finite dimensional simple n -Lie algebra for $n > 2$ over an algebraically closed field of characteristic zero [14], which is the $(n+1)$ -dimensional n -Lie algebra. So far, the only known infinite dimensional simple n -Lie algebras over fields of characteristic $p \geq 0$ are Jacobian algebras and their

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quotient algebras [15, 16]. The first author of the current paper and her collaborators [17] showed that there exist only $\lfloor \frac{n}{2} \rfloor + 1$ classes of $(n+1)$ -dimensional simple n -Lie algebras over a complete field of characteristic 2. They also showed that there are no simple $(n+2)$ -dimensional n -Lie algebras.

In [22], 6-dimensional 4-Lie algebras were classified and some basic properties of $(n+2)$ -dimensional n -Lie algebras were studied. The purpose of this paper is to classify the $(n+2)$ -dimensional n -Lie algebras over an algebraically closed field of characteristic zero. Our results are expected to be useful in various applications.

The organization for the rest of this paper is as follows. Section 2 introduces some basic notions. Section 3 is devoted to the properties and classification of the $(n+2)$ -dimensional n -Lie algebras.

2. Fundamental notions

An n -Lie algebra is a vector space A over a field F ($\text{char}(F) \neq 2$) equipped with an n -multilinear operation $[x_1, \dots, x_n]$ satisfying

$$[x_1, \dots, x_n] = \text{sgn}(\sigma)[x_{\sigma(1)}, \dots, x_{\sigma(n)}], \quad (2.1)$$

and

$$[[x_1, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, [x_i, y_2, \dots, y_n], \dots, x_n] \quad (2.2)$$

for any $x_1, \dots, x_n, y_2, \dots, y_n \in A$ and any permutation $\sigma \in S_n$. Identity (2.2) is usually called the generalized Jacobi identity, or simply the Jacobi identity.

A derivation of an n -Lie algebra A is a linear map D of A into itself satisfying

$$D([x_1, \dots, x_n]) = \sum_{i=1}^n [x_1, \dots, D(x_i), \dots, x_n] \quad (2.3)$$

for any $x_1, \dots, x_n \in A$. Let $\text{Der}(A)$ be the set of all derivations of A . Then $\text{Der}(A)$ is a Lie subalgebra of the general linear Lie algebra $gl(A)$ and is called the derivation algebra of A . The map $\text{ad}(x_1, \dots, x_{n-1}): A \rightarrow A$, given by

$$\text{ad}(x_1, \dots, x_{n-1})(x_n) = [x_1, \dots, x_n], \text{ for } x_n \in A,$$

is referred to as a left multiplication defined by elements $x_1, \dots, x_{n-1} \in A$. It follows from identity (2.2), that $\text{ad}(x_1, \dots, x_{n-1})$ is a derivation. The set of all finite linear combinations of left multiplications is an ideal of $\text{Der}(A)$, which we denote by $\text{ad}(A)$. Every derivation in $\text{ad}(A)$ is by definition an inner derivation.

If a subspace B of an n -Lie algebra A satisfying $[x_1, \dots, x_n] \in B$ for any $x_1, \dots, x_n \in B$, then B is called a subalgebra of A . Let A_1, A_2, \dots, A_n be subalgebras of an n -Lie algebra A . Denote by $[A_1, A_2, \dots, A_n]$ the subspace of A generated by all vectors $[x_1, \dots, x_n]$, where $x_i \in A_i$ for $i = 1, 2, \dots, n$. The subalgebra $A^1 = [A, A, \dots, A]$ is called the derived algebra of A . If $A^1 = 0$, then A is called an abelian n -Lie algebra.

Let H be an abelian subalgebra of n -Lie algebra A . Then H is by definition a Toral subalgebra of A , if A is a complete H -module, that is

$$A = \bigoplus_{\alpha \in (H^{n-1})^*} A_\alpha \quad (\text{direct sum as vector spaces}),$$

where

$$A_\alpha = \{x \in A \mid \text{ad}(h_1, \dots, h_{n-1})(x) = \alpha(h_1, \dots, h_{n-1})(x), \forall (h_1, h_2, \dots, h_{n-1}) \in H^{n-1}\}.$$

A Toral subalgebra H is called maximal if there are no Toral subalgebras of A properly containing H . An ideal I of an n -Lie algebra A is a subspace of A such that $[I, A, \dots, A] \subseteq I$. If $[I, I, A, \dots, A] = 0$, then I is referred to as an abelian ideal. If $A^1 \neq 0$ and A has no ideals except 0 and itself, then A is by definition a simple n -Lie algebra. An n -Lie algebra A is said to be decomposable if there are nonzero ideals I_1, I_2 such that

$$A = I_1 \oplus I_2,$$

then $[I_1, I_2, A, \dots, A] = 0$. Otherwise, we say that A is indecomposable. Clearly if A is a simple n -Lie algebra then A is indecomposable.

The subset $Z(A) = \{x \in A \mid [x, y_1, \dots, y_{n-1}] = 0, \forall y_1, \dots, y_{n-1} \in A\}$ is called the center of A . It is clear that $Z(A)$ is an abelian ideal of A .

3. Classification of $(n+2)$ -dimensional n -Lie algebras

In this section, unless stated otherwise, we suppose that F is an algebraically closed field of characteristic 0. Any brackets of basis vectors not listed in the multiplication table of n -Lie algebras are assumed to be zero.

First, we prove the isomorphic criterion theorem for $(n+2)$ -dimensional n -Lie algebras over F .

We need some symbols for reducing our description. Suppose $[\dots,]_1$ and $[\dots,]_2$ are two n -ary Lie products on vector space A such that $(A, [\dots,]_1)$ and $(A, [\dots,]_2)$ are n -Lie algebras. Let e_1, e_2, \dots, e_{n+2} be a basis of A . Set

$$e_{i,j} = [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}]_1 = \sum_{k=1}^{n+2} b_{i,j}^k e_k, \quad b_{i,j}^k \in F, 1 \leq i < j \leq n+2, \quad (3.1)$$

then

$$(e_{1,2}, e_{1,3}, \dots, e_{1,n+2}, e_{2,3}, \dots, e_{2,n+2}, \dots, e_{n+1,n+2}) = (e_1, e_2, \dots, e_{n+2})B,$$

where

$$B = \begin{pmatrix} b_{1,2}^1 & b_{1,3}^1 & \cdots & b_{1,n+2}^1 & b_{2,3}^1 & \cdots & b_{n+1,n+2}^1 \\ b_{1,2}^2 & b_{1,3}^2 & \cdots & b_{1,n+2}^2 & b_{2,3}^2 & \cdots & b_{n+1,n+2}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{1,2}^{n+2} & b_{1,3}^{n+2} & \cdots & b_{1,n+2}^{n+2} & b_{2,3}^{n+2} & \cdots & b_{n+1,n+2}^{n+2} \end{pmatrix}, \quad b_{i,j}^k \in F, 1 \leq i < j \leq n+2.$$

Then the multiplication of $(A, [\dots,]_1)$ is determined by the $((n+2) \times \frac{(n+1)(n+2)}{2})$ matrix B . And B is called the structure matrix of $(A, [\dots,]_1)$ with respect to the basis e_1, e_2, \dots, e_{n+2} .

Similarly denote \bar{B} is the structure matrix of $(A, [\dots,]_2)$ with respect to the basis e_1, e_2, \dots, e_{n+2} , that is

$$\bar{e}_{ij} = [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}]_2 = \sum_{k=1}^{n+2} \bar{b}_{i,j}^k e_k, \quad \bar{b}_{i,j}^k \in F, 1 \leq i < j \leq n+2, \quad (3.2)$$

$$(\bar{e}_{1,2}, \bar{e}_{1,3}, \dots, \bar{e}_{1,n+2}, \bar{e}_{2,3}, \dots, \bar{e}_{2,n+2}, \dots, \bar{e}_{n+1,n+2}) = (e_1, \dots, e_{n+2})\bar{B}.$$

Theorem 3.1. N -Lie algebras $(A, [\dots]_1)$ and $(A, [\dots]_2)$ with products (3.1) and (3.2) on an $(n+2)$ -dimensional linear space A are isomorphic if and only if there exists a nonsingular $((n+2) \times (n+2))$ matrix $T = (t_{i,j})$ such that

$$B = T'^{-1}\bar{B}T_*, \quad (3.3)$$

where T' is the transpose matrix of T , and $T_* = (T_{k,l}^{i,j})$ is an $(\frac{(n+1)(n+2)}{2} \times \frac{(n+1)(n+2)}{2})$ matrix, and $T_{k,l}^{i,j} \in F$ is the determinant defined by (3.5) below for $1 \leq i, j, k, l \leq n+2$.

Proof. If n -Lie algebra $(A, [\dots]_1)$ is isomorphic to $(A, [\dots]_2)$ under the isomorphism σ . Let e_1, \dots, e_{n+2} be a basis of A , and structural matrices are (3.1) and (3.2) with respect to e_1, \dots, e_{n+2} respectively, that is

$$e_{i,j} = [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}]_1 = \sum_{k=1}^{n+2} b_{i,j}^k e_k, \quad B = (b_{i,j}^k)_{(n+2) \times \frac{(n+2)(n+2)}{2}};$$

and

$$\bar{e}_{i,j} = [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}]_2 = \sum_{k=1}^{n+2} \bar{b}_{i,j}^k e_k, \quad \bar{B} = (\bar{b}_{i,j}^k)_{(n+2) \times \frac{(n+2)(n+2)}{2}}.$$

Denote $e'_i = \sigma(e_i)$, $1 \leq i \leq n+2$ and the nonsingular $((n+2) \times (n+2))$ matrix $T = (t_{ij})$ is the transition matrix of σ in the basis e_1, e_2, \dots, e_{n+2} , that is

$$(\sigma(e_1), \dots, \sigma(e_{n+2})) = (e'_1, \dots, e'_{n+2}) = (e_1, e_2, \dots, e_{n+2})T. \quad (3.4)$$

Then

$$\begin{aligned} e'_{k,l} &= [e'_1, \dots, \hat{e}'_k, \dots, \hat{e}'_l, \dots, e'_{n+2}]_2 \\ &= [\sum_{m=1}^{n+2} t_{m,1} e_m, \sum_{m=1}^{n+2} t_{m,2} e_m, \dots, \sum_{m=1}^{n+2} t_{m,k-1} e_m, \sum_{m=1}^{n+2} t_{m,k+1} e_m, \\ &\quad \dots, \sum_{m=1}^{n+2} t_{m,l-1} e_m, \sum_{m=1}^{n+2} t_{m,l+1} e_m, \dots, \sum_{m=1}^{n+2} t_{m,n+2} e_m]_2 \\ &= T_{k,l}^{1,2} \bar{e}_{1,2} + T_{k,l}^{1,3} \bar{e}_{1,3} + \dots + T_{k,l}^{1,n+2} \bar{e}_{1,n+2} + T_{k,l}^{2,3} \bar{e}_{2,3} + \dots + T_{k,l}^{n+1,n+2} \bar{e}_{n+1,n+2}, \end{aligned}$$

where

$$T_{k,l}^{i,j} = \det \begin{pmatrix} t_{1,1} & \cdots & t_{1,k-1} & t_{1,k+1} & \cdots & t_{1,l-1} & t_{1,l+1} & \cdots & t_{1,n+2} \\ t_{2,1} & \cdots & t_{2,k-1} & t_{2,k+1} & \cdots & t_{2,l-1} & t_{2,l+1} & \cdots & t_{2,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{i-1,1} & \cdots & t_{i-1,k-1} & t_{i-1,k+1} & \cdots & t_{i-1,l-1} & t_{i-1,l+1} & \cdots & t_{i-1,n+2} \\ t_{i+1,1} & \cdots & t_{i+1,k-1} & t_{i+1,k+1} & \cdots & t_{i+1,l-1} & t_{i+1,l+1} & \cdots & t_{i+1,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{j-1,1} & \cdots & t_{j-1,k-1} & t_{j-1,k+1} & \cdots & t_{j-1,l-1} & t_{j-1,l+1} & \cdots & t_{j-1,n+2} \\ t_{j+1,1} & \cdots & t_{j+1,k-1} & t_{j+1,k+1} & \cdots & t_{j+1,l-1} & t_{j+1,l+1} & \cdots & t_{j+1,n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{n+1,1} & \cdots & t_{n+1,k-1} & t_{n+1,k+1} & \cdots & t_{n+1,l-1} & t_{n+1,l+1} & \cdots & t_{n+1,n+2} \\ t_{n+2,1} & \cdots & t_{n+2,k-1} & t_{n+2,k+1} & \cdots & t_{n+2,l-1} & t_{n+2,l+1} & \cdots & t_{n+2,n+2} \end{pmatrix} \quad (3.5)$$

$1 \leq i < j \leq n+2$, $1 \leq k \neq l \leq n+2$. Denote

$$T_* = \begin{pmatrix} T_{1,2}^{1,2} & T_{1,3}^{1,2} & \cdots & T_{1,n+2}^{1,2} & T_{2,3}^{1,2} & \cdots & T_{n+1,n+2}^{1,2} \\ T_{1,2}^{1,3} & T_{1,3}^{1,3} & \cdots & T_{1,n+2}^{1,3} & T_{2,3}^{1,3} & \cdots & T_{n+1,n+2}^{1,3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ T_{1,2}^{n,n+2} & T_{1,3}^{n,n+2} & \cdots & T_{1,n+2}^{n,n+2} & T_{2,3}^{n,n+2} & \cdots & T_{n+1,n+2}^{n,n+2} \\ T_{1,2}^{n+1,n+2} & T_{1,3}^{n+1,n+2} & \cdots & T_{1,n+2}^{n+1,n+2} & T_{2,3}^{n+1,n+2} & \cdots & T_{n+1,n+2}^{n+1,n+2} \end{pmatrix}, \quad (3.6)$$

then T_* is a $(\frac{(n+1)(n+2)}{2} \times \frac{(n+1)(n+2)}{2})$ matrix, and

$$(e'_{1,2}, e'_{1,3}, \dots, e'_{1,n+2}, e'_{2,3}, \dots, e'_{n+1,n+2}) = (\bar{e}_{1,2}, \bar{e}_{1,3}, \dots, \bar{e}_{1,n+2}, \bar{e}_{2,3}, \dots, \bar{e}_{n+1,n+2}) T_*. \quad (3.7)$$

From identities (3.1) and (3.2) that

$$(e'_{1,2}, e'_{1,3}, \dots, e'_{1,n+2}, e'_{2,3}, \dots, e'_{n+1,n+2}) = (e_1, e_2, \dots, e_{n+2}) \bar{B} T_*. \quad (3.8)$$

Furthermore

$$\begin{aligned} e'_{k,l} &= [e'_1, \dots, \hat{e}'_k, \dots, \hat{e}'_l, \dots, e'_{n+2}]_2 = [\sigma(e_1), \dots, \sigma(\hat{e}_k), \dots, \sigma(\hat{e}_l), \dots, \sigma(e_{n+2})]_2 \\ &= \sigma([e_1, \dots, \hat{e}_k, \dots, \hat{e}_l, \dots, e_{n+2}]_1) = \sigma(e_{kl}) = \sum_{i=1}^{n+2} b_{kl}^i \sigma(e_i) = \sum_{s=1}^{n+2} \left(\sum_{i=1}^{n+2} b_{kl}^i t_{si} \right) e_s. \end{aligned}$$

Thus

$$(e'_{1,2}, e'_{1,3}, \dots, e'_{1,n+2}, e'_{2,3}, \dots, e'_{n+1,n+2}) = (e_1, e_2, \dots, e_{n+2}) T' B. \quad (3.9)$$

It follows from (3.8) and (3.9) that

$$T' B = \bar{B} T_*, \text{ that is } B = T'^{-1} \bar{B} T_*.$$

On the other hand, we take a linear transformation σ of A , such that $\sigma(e_1, \dots, e_{n+2}) = (e_1, \dots, e_{n+2}) T$. By similar discussions to the above we have σ is an n -Lie isomorphism from $(A, [\dots,]_1)$ to $(A, [\dots,]_2)$. \square

It is complex when we use Theorem 3.1 to judge the isomorphism of two $(n+2)$ -dimensional n -Lie algebras due to the massive computations involved. But from (3.5) and (3.3), the computation is orderly so it is easy to use computer.

Before giving the classification theorem, we need to classify the $(n+1)$ -dimensional n -Lie algebras first.

Lemma 3.1. Let A be an $(n+1)$ -dimensional n -Lie algebra over F and e_1, e_2, \dots, e_{n+1} be a basis of A ($n \geq 3$). Then one and only one of the following possibilities holds up to isomorphisms:

(a) If $\dim A^1 = 0$, then A is an abelian n -Lie algebra.

(b) If $\dim A^1 = 1$ and let $A^1 = Fe_1$, then in the case that $A^1 \subseteq Z(A)$,

$$(b_1) [e_2, \dots, e_{n+1}] = e_1; \quad (3.10)$$

in the case that A^1 is not contained in $Z(A)$,

$$(b_2) [e_1, \dots, e_n] = e_1. \quad (3.11)$$

(c) If $\dim A^1 = 2$ and let $A^1 = Fe_1 + Fe_2$, then

$$(c_1) \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2; \end{cases} \quad (c_2) \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2; \end{cases}$$

$$(c_3) \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, \dots, e_{n+1}] = e_2, \end{cases} \quad (3.12)$$

where $\alpha \in F$ and $\alpha \neq 0$.

(d) If $\dim A^1 = r$, $3 \leq r \leq n+1$, let $A^1 = Fe_1 + Fe_2 + \dots + Fe_r$. Then

$$(d_r) [e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = e_i, \quad 1 \leq i \leq r, \quad (3.13)$$

where symbol \hat{e}_i means that e_i is omitted.

Proof. If $\dim A^1 = 1$ or $\dim A^1 > 2$, the classification has been discussed by [3]. Now we study the case $\dim A^1 = 2$. Set $A^1 = Fe_n + Fe_{n+1}$, $e^i = (-1)^{n+1+i}[e_1, \dots, \hat{e}_i, \dots, e_{n+1}] = \beta_{ni}e_n + \beta_{n+1i}e_{n+1}$, $1 \leq i \leq n-1$. Then we have

$$(c) \begin{cases} e^1 = (-1)^{n+1+1}[e_2, \dots, e_{n+1}] = \beta_{11}e_1 + \beta_{21}e_2, \\ e^2 = (-1)^{n+1+2}[e_1, e_3, \dots, e_{n+1}] = \beta_{12}e_1 + \beta_{22}e_2, \\ e^i = (-1)^{n+1+i}[e_1, e_2, e_3, \dots, \hat{e}_i, \dots, e_{n+1}] = \beta_{1i}e_1 + \beta_{2i}e_2, \quad (3 \leq i \leq n+1). \end{cases}$$

and

$$\begin{vmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{vmatrix} \neq 0, \quad (\beta_{i2} - \beta_{2i})e^1 + (\beta_{1i} - \beta_{i1})e^2 + (\beta_{21} - \beta_{12})e^i = 0. \quad (*)$$

It follows that e^i and e^j are linearly dependent for $i \neq j$ for $i, j = 3, \dots, n+1$. And by $\dim A^1 = 2$ and $(*)$, we have $e^i = 0$ for $3 \leq i \leq n+1$. Then (c) is reduced to

$$(1) \begin{cases} e^1 = (-1)^{n+1+1}[e_2, \dots, e_{n+1}] = ae_1 + ce_2, \\ e^2 = (-1)^{n+1+2}[e_1, e_3, \dots, e_{n+1}] = be_1 + de_2. \end{cases} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \det A \neq 0.$$

If $a \neq 0$, let $P = \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ \frac{-c}{\sqrt{(ad-bc)a}} & \frac{\sqrt{a}}{\sqrt{ad-bc}} \end{pmatrix}$, then

$$(\det P^{-1})PAP' = \begin{pmatrix} \sqrt{ad-bc} & \frac{b-c}{\sqrt{ad-bc}} \\ 0 & \sqrt{ad-bc} \end{pmatrix}.$$

By Theorem 2 in [3], (1) is isomorphic to

$$(1)' \quad \begin{cases} e^1 = (-1)^{n+1+1}[e_2, \dots, e_{n+1}] = \sqrt{ad-bc}e_1, \\ e^2 = (-1)^{n+1+2}[e_1, e_3, \dots, e_{n+1}] = (b-c)e_1 + \sqrt{ad-bc}e_2, \end{cases}$$

In the case of $b-c=0$, substituting e_1 and e_{n+1} by ie_1 and $(-1)^{n+1+1}\frac{1}{\sqrt{ad-bc}}ie_{n+1}$ respectively, we get (1) is isomorphic to

$$(c_1) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2. \end{cases}$$

In the case of $b-c \neq 0$, substituting e_1 and e_{n+1} by $e_1 + \frac{(b-c)}{\sqrt{ad-bc}}e_2$ and $\frac{e_{n+1}}{\sqrt{ad-bc}}$ we get (1) is isomorphic to

$$\begin{cases} [e_2, \dots, e_{n+1}] = (-1)^{n+1+1}e_1 - (-1)^{n+1+1}\frac{(b-c)}{\sqrt{ad-bc}}e_2, \\ [e_1, e_3, \dots, e_{n+1}] = (-1)^{n+1+2}e_2, \end{cases}$$

substituting e_1, e_2 and e_{n+1} by ie_1 and $-\frac{\sqrt{ad-bc}}{b-c}ie_2$, $(-1)^{n+1+1}\frac{\sqrt{ad-bc}}{b-c}e_{n+1}$ we get (1) is isomorphic to

$$(c_2) \quad \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \end{cases}$$

where $\alpha = -\frac{ad-bc}{(b-c)^2} \neq 0$.

If $a = 0$, then (1) is of the form

$$\begin{cases} e^1 = (-1)^{n+1+1}[e_2, \dots, e_{n+1}] = ce_2, \\ e^2 = (-1)^{n+1+2}[e_1, e_3, \dots, e_{n+1}] = be_1 + de_2. \end{cases} \quad \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \neq 0.$$

In the cases of $d \neq 0$ or $b+c \neq 0$, by the similar discussion to above, we have (1) is isomorphic to the case (c_1) or (c_2) . In the case of $d = b+c = 0$, (1) is isomorphic to

$$(c_3) \quad \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, \dots, e_{n+1}] = e_2. \end{cases}$$

□

Lemma 3.2.[22] Let A be a nonabelian $(n+2)$ -dimensional n -Lie algebra over F . If $\dim A^1 \neq 3$, then there exists a non-abelian subalgebra of codimension 1 containing A^1 .

Lemma 3.3.[22] Let A be an $(n+2)$ -dimensional n -Lie algebra over F . Then we have $\dim A^1 \leq n+1$.

Theorem 3.2. Let A be an $(n+2)$ -dimensional n -Lie algebra over F with a basis e_1, \dots, e_{n+2} . Then one and only one of the following possibilities holds up to isomorphisms:

(a) If $\dim A^1 = 0$, then A is an abelian n -Lie algebra.

(b) If $\dim A^1 = 1$, let $A^1 = Fe_1$. Then we have

(b¹) in the case that $A^1 \subseteq Z(A)$, $[e_2, \dots, e_{n+1}] = e_1$;

(b²) in the case that A^1 is not contained in $Z(A)$, $[e_1, \dots, e_n] = e_1$.

(c) If $\dim A^1 = 2$, let $A^1 = Fe_1 + Fe_2$. Then we have

$$\begin{aligned}
(c^1) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = e_2; \end{array} \right. & (c^2) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1; \end{array} \right. \\
(c^3) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2; \end{array} \right. & (c^4) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1; \end{array} \right. \\
(c^5) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2; \end{array} \right. & (c^6) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1; \end{array} \right. \\
(c^7) \left\{ \begin{array}{l} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2; \end{array} \right.
\end{aligned}$$

where $\alpha \in F$, and $\alpha \neq 0$.

(d) If $\dim A^1 = 3$, let $A^1 = Fe_1 + Fe_2 + Fe_3$. Then we have

$$\begin{aligned}
(d^1) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = -e_2, \\ [e_3, \dots, e_{n+2}] = e_3; \end{array} \right. & (d^2) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = e_3 + \alpha e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_1, e_4, \dots, e_{n+2}] = e_1; \end{array} \right. \\
(d^3) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, e_4, \dots, e_{n+2}] = e_3, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = 2e_1; \end{array} \right. & (d^4) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, e_2, e_4, \dots, e_{n+1}] = e_3; \end{array} \right. \\
(d^5) \left\{ \begin{array}{l} [e_1, e_4, \dots, e_{n+2}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_3, e_4, \dots, e_{n+2}] = \beta e_2 + (1 + \beta)e_3, \quad \beta \in F, \beta \neq 0, 1; \end{array} \right. \\
(d^6) \left\{ \begin{array}{l} [e_1, e_4, \dots, e_{n+2}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_3, e_4, \dots, e_{n+2}] = e_3; \end{array} \right.
\end{aligned}$$

$$(d^7) \begin{cases} [e_1, e_4, \dots, e_{n+2}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_3, e_4, \dots, e_{n+2}] = se_1 + te_2 + ue_3, \quad s, t, u \in F, \quad s \neq 0. \end{cases}$$

And n -Lie algebras corresponding to the case (d^7) with coefficients s, t, u and s', t', u' are isomorphic if and only if there exists a nonzero element $r \in F$ such that

$$s = r^3 s', \quad t = r^2 t', \quad u = ru', \quad s, s', t, t', u, u' \in F.$$

(r) If $\dim A^1 = r, 4 \leq r \leq n+1$, let $A^1 = Fe_1 + \dots + Fe_r$. Then we have

$$(r^1) \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = e_2, \\ \dots \dots \dots \dots, \\ [e_2, \dots, \hat{e}_i, \dots, e_r, \dots, e_{n+2}] = e_i, \\ \dots \dots \dots \dots, \\ [e_2, \dots, e_{r-1}, e_{r+1}, \dots, e_{n+2}] = e_r; \end{cases} \quad (r^2) \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ \dots \dots \dots \dots, \\ [e_1, \dots, \hat{e}_i, \dots, e_r, \dots, e_{n+1}] = e_i, \\ \dots \dots \dots \dots, \\ [e_1, \dots, e_{r-1}, e_{r+1}, \dots, e_{n+1}] = e_r. \end{cases}$$

Proof. 1. Case (a) is trivial.

2. Case (b). Suppose $A^1 = Fe_1$. Then from Lemma 3.1, Lemma 3.2 and Lemma 3.3, the multiplication table of A in the basis e_1, \dots, e_{n+2} has the following possibilities

$$(1) \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}e_1; \end{cases}$$

$$(2) \begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}e_1, \end{cases}$$

where $b_{ij} \in F, 1 \leq i < j \leq n+1$.

Firstly, substituting the first identity of (1) into its other equations and using the Jacobi identities, we get

$$\begin{aligned} b_{ij}e_1 &= [e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\ &= [[e_2, \dots, e_{n+1}], e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\ &= [e_2, \dots, [e_i, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}], \dots, e_{n+1}] \\ &\quad + [e_2, \dots, [e_j, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}], \dots, e_{n+1}] \\ &= [e_2, \dots, (-1)^{i-2}b_{1j}e_1, \dots, e_{n+1}] + [e_2, \dots, (-1)^{j-3}b_{1i}e_1, \dots, e_{n+1}] \\ &= b_{1j}[e_1, e_2, \dots, \hat{e}_i, \dots, e_{n+1}] + b_{1i}[e_1, e_2, \dots, \hat{e}_j, \dots, \dots e_{n+1}] = 0, \quad 2 \leq i < j \leq n+1. \end{aligned}$$

Then (1) is in the form of

$$(1)' \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_2, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{1j}e_1, \quad 2 \leq j \leq n+1. \end{cases}$$

Replacing e_{n+2} by $e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}e_j$ in (1)', we get that (1) is isomorphic to

$$(b^1) [e_2, \dots, e_{n+1}] = e_1.$$

By similar discussion we get that (2) is isomorphic to (b^2) . And (b^1) is not isomorphic to (b^2) since (b^1) has a nonzero center.

3. If $\dim A^1 = 2$, suppose $A^1 = Fe_1 + Fe_2$. By Lemma 3.1, Lemma 3.2 and Lemma 3.3, the multiplication table in the basis e_1, \dots, e_{n+2} has the following possibilities

$$\begin{aligned}
(1) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2; \end{cases} \\
(2) \quad & \begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2; \end{cases} \\
(3) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2; \end{cases} \\
(4) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2, \end{cases} \\
(5) \quad & \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2; \end{cases}
\end{aligned}$$

where $b_{ij} \in F$, $1 \leq i < j \leq n+1$.

Firstly imposing the Jacobi identities on (1) we get

$$\begin{aligned}
& b_{ij}^1 e_1 + b_{ij}^2 e_2 = [e_1, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\
& = [[e_2, \dots, e_{n+1}], e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\
& = [e_2, \dots, [e_i, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}], \dots, e_{n+1}] \\
& + [e_2, \dots, [e_j, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}], \dots, e_{n+1}] \\
& = [e_2, \dots, (-1)^{i-2}(b_{1j}^1 e_1 + b_{1j}^2 e_2), \dots, e_{n+1}] + [e_2, \dots, (-1)^{j-3}(b_{1i}^1 e_1 + b_{1i}^2 e_2), \dots, e_{n+1}] \\
& = 0, \text{ for } 3 \leq i < j \leq n+1.
\end{aligned}$$

When $i = 2$ and $3 \leq j \leq n+1$,

$$\begin{aligned}
& b_{2j}^1 e_1 + b_{2j}^2 e_2 = [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\
& = [[e_2, \dots, e_{n+1}], e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\
& = [b_{1j}^1 e_1 + b_{1j}^2 e_2, e_3, \dots, e_{n+1}] + [e_2, \dots, (-1)^{j-3}(b_{12}^1 e_1 + b_{12}^2 e_2), \dots, e_{n+1}] = b_{1j}^2 e_1.
\end{aligned}$$

And again replacing e_{n+2} by $e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}^1 e_j$, we get

$$(1)' \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = b_{12}^2 e_2, \\ [e_2, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{1j}^2 e_2, \\ [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{1j}^2 e_1, \end{cases} \quad 3 \leq j \leq n+1.$$

If $b_{12}^2 \neq 0$, $b_{1j}^2 = 0$, $3 \leq j \leq n+1$, substituting $\frac{e_{n+2}}{b_{12}^2}$ for e_{n+2} in (1)', we get (1) is isomorphic to

$$(c^1) \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = e_2. \end{cases}$$

If there exists j such that $b_{1j}^2 \neq 0, 3 \leq j \leq n+1$, then we might as well suppose $b_{13}^2 \neq 0$.

Substituting $e_3 + \sum_{j=4}^{n+1} (-1)^{j-3} \frac{b_{1j}^2}{b_{13}^2} e_j - \frac{b_{12}^2}{b_{13}^2} e_2$ for e_3 and $\frac{e_{n+2}}{b_{13}^2}$ for e_{n+2} in (1)', we get

$$(c^2) \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1. \end{cases}$$

Secondly, substituting $e_{n+2} - \sum_{i=1}^n (-1)^{n-i} b_{in+1}^1 e_i$ for e_{n+2} in (2), we get

$$\begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2, \quad 1 \leq i < j \leq n, \\ [e_1, \dots, \hat{e}_i, \dots, e_n, e_{n+2}] = b_{in+1}^2 e_2, \quad 1 \leq i \leq n. \end{cases}$$

Since $b_{2n+1}^2 e_2 = [e_1, e_3, \dots, e_n, e_{n+2}] = [[e_1, \dots, e_n], e_3, \dots, e_n, e_{n+2}]$

$= [b_{2n+1}^2 e_2, e_2, \dots, e_n] + [e_1, b_{1n+1}^2 e_2, e_3, \dots, e_n] = b_{1n+1}^2 e_1$, and

$b_{in+1}^2 e_2 = [e_1, e_2, e_3, \dots, \hat{e}_i, \dots, e_n, e_{n+2}] = [[e_1, \dots, e_n], e_2, e_3, \dots, \hat{e}_i, \dots, e_n, e_{n+2}]$

$= [b_{in+1}^2 e_2, e_2, \dots, e_n] + [e_1, e_2, e_3, \dots, (-1)^{i-2} b_{1n+1}^2 e_2, \dots, e_n] = 0$,

we have $b_{2n+1}^2 = b_{1n+1}^2 = 0$ and $b_{in+1}^2 = 0, 3 \leq i \leq n$. Then (2) is isomorphic to

$$\begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2, \quad 1 \leq i < j \leq n. \end{cases}$$

When $i = 1, 2 \leq j \leq n$, since

$$0 = [[e_1, e_3, \dots, e_n, e_{n+2}], e_2, \dots, \hat{e}_j, \dots, e_n, e_{n+1}]$$

$$= [e_1, e_3, \dots, e_n, [e_{n+2}, e_2, \dots, \hat{e}_j, \dots, e_n, e_{n+1}]]$$

$$= (-1)^{2n-3} b_{1j}^2 e_1, \text{ we have } b_{1j}^2 = 0, \quad 2 \leq j \leq n.$$

If $i = 2, 3 \leq j \leq n$, by

$$b_{2j}^1 e_1 + b_{2j}^2 e_2 = [e_1, e_3, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}]$$

$$= [[e_1, \dots, e_n], e_3, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}]$$

$$= [b_{2j}^1 e_1 + b_{2j}^2 e_2, e_2, \dots, e_n] + [e_1, b_{1j}^1 e_1 + b_{1j}^2 e_2, e_3, \dots, e_n]$$

$$+ [e_1, e_2, e_3, \dots, [e_j, e_3, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}], \dots, e_n]$$

$$= b_{2j}^1 e_1 + b_{1j}^2 e_1 = b_{2j}^1 e_1,$$

we obtain $b_{2j}^2 = 0, 3 \leq j \leq n$. If $3 \leq i < j \leq n$, by

$$b_{ij}^1 e_1 + b_{ij}^2 e_2 = [e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}]$$

$$= [[e_1, \dots, e_n], e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}]$$

$$= [b_{ij}^1 e_1 + b_{ij}^2 e_2, e_2, \dots, e_n] + [e_1, e_2, \dots, [e_i, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}], \dots, e_n]$$

$$+[e_1, e_2, \dots, [e_j, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, e_{n+1}, e_{n+2}], \dots, e_n] = b_{ij}^1 e_1,$$

we get $b_{ij}^2 = 0, 3 \leq i < j \leq n$. Then (2) is isomorphic to

$$\begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1, \quad 1 \leq i < j \leq n. \end{cases}$$

This contradicts $\dim A^1 = 2$. Therefore, table (2) is not realized.

Thirdly we study the case (3). For $i = 1, 3 \leq j \leq n+1$, since

$$b_{1j}^1 e_1 + b_{1j}^2 e_2 = [e_2, \dots, \hat{e}_j, \dots, e_{n+2}] = [[e_1, e_3, \dots, e_{n+1}], \dots, \hat{e}_j, \dots, e_{n+2}] = b_{2j}^1 e_2 + b_{2j}^2 e_1, \text{ we have } b_{2j}^1 = b_{1j}^2, b_{2j}^2 = b_{1j}^1, \quad 3 \leq j \leq n+1. \text{ For } 3 \leq i < j \leq n+1, \text{ from}$$

$$\begin{aligned} b_{ij}^1 e_1 + b_{ij}^2 e_2 &= [e_1, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\ &= [[e_2, \dots, e_{n+1}], e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\ &= [e_2, \dots, [e_i, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}], \dots, e_{n+1}] \\ &+ [e_2, \dots, [e_j, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}], \dots, e_{n+1}] = 0, \end{aligned}$$

we have $b_{ij}^1 = b_{ij}^2 = 0, 3 \leq i < j \leq n+1$. Again substituting $e_{n+2} + \sum_{j=2}^{n+1} (-1)^{n+2-j} b_{1j}^1 e_j + (-1)^n b_{12}^2 e_1$ for e_{n+2} , (3) is isomorphic to

$$(3)' \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_2, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{1j}^2 e_2, \\ [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{1j}^2 e_1, \end{cases} \quad 3 \leq j \leq n+1.$$

If $b_{1j}^2 = 0, 3 \leq j \leq n+1$, then (3) is isomorphic to

$$(c^3) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2. \end{cases}$$

If there exists $b_{1j}^2 \neq 0$ for $3 \leq j \leq n+1$, then we might as well suppose $b_{13}^2 \neq 0$.

Replacing e_3 and e_{n+2} by $e_3 + \sum_{j=4}^{n+1} (-1)^{j-3} \frac{b_{1j}^2}{b_{13}^2} e_j$ and $\frac{e_{n+2}}{b_{13}^2}$ in (3)' respectively, we get (3) is of the form

$$(c^4) \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1. \end{cases}$$

Fourthly, we study the case (4). For $i = 1, 3 \leq j \leq n+1$, by $b_{1j}^1 e_1 + b_{1j}^2 e_2 = [e_2, \dots, \hat{e}_j, \dots, e_{n+2}] = [[e_1, e_3, \dots, e_{n+1}], \dots, \hat{e}_j, \dots, e_{n+2}] = b_{2j}^1 e_2 + b_{2j}^2 \alpha e_1 + b_{2j}^2 e_2$, we have $b_{1j}^1 = b_{2j}^2 \alpha, b_{1j}^2 = b_{2j}^1 + b_{2j}^2, 3 \leq j \leq n+1$.

For $3 \leq i < j \leq n+1$, we have $b_{ij}^1 = b_{ij}^2 = 0, 3 \leq i < j \leq n+1$ since

$$\begin{aligned} b_{ij}^1 e_1 + b_{ij}^2 e_2 &= [e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}] \\ &= \frac{1}{\alpha} [[e_2, \dots, e_{n+1}] - e_2, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\alpha} [e_2, \dots, [e_i, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}], \dots, e_{n+1}] \\
&+ \frac{1}{\alpha} [e_2, \dots, [e_j, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}], \dots, e_{n+1}] = 0.
\end{aligned}$$

Then if substituting $e_{n+2} + \sum_{j=3}^{n+1} (-1)^{n+2-j} b_{2j}^2 e_j + (-1)^n b_{12}^2 e_1$ for e_{n+2} in (4), we get

$$(4)' \quad \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_3, \dots, e_{n+2}] = b_{12}^1 e_1, \\ [e_2, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{2j}^1 e_2, \\ [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{2j}^1 e_1, \end{cases} \quad 3 \leq j \leq n+1.$$

If $b_{12}^1 = b_{2j}^1 = 0, 3 \leq j \leq n+1$, (4) is isomorphic to

$$(c^5) \quad \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2. \end{cases}$$

If $b_{12}^1 \neq 0, b_{2j}^1 = 0, 3 \leq j \leq n+1$, replacing e_{n+2} by $(-1)^n \frac{\alpha}{b_{12}^1} e_{n+2} - e_1 + e_2$ in (4)', we get (c⁵).

If there exists $b_{2j}^1 \neq 0$, for some $3 \leq j \leq n+1$. We might as well suppose $b_{23}^1 \neq 0$. Substituting $e_3 + \sum_{j=4}^{n+1} (-1)^{j-3} \frac{b_{2j}^1}{b_{23}^1} e_j - \frac{b_{12}^1}{b_{23}^1} e_1$ and $\frac{e_{n+2}}{b_{23}^1}$ for e_3 and e_{n+2} in (4)' respectively, we get

$$(c^6) \quad \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = e_1. \end{cases}$$

Lastly, we study the case (5). For $3 \leq i < j \leq n+1$, we have $b_{ij}^1 = b_{ij}^2 = 0$ since

$$\begin{aligned}
&[e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}]. \\
&= [[e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}], e_3, \dots, e_{n+1}] \\
&= [e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}] + [e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+2}]
\end{aligned}$$

Then if substituting $e_{n+2} + (-1)^n b_{12}^1 e_1 - \sum_{i=1}^{n+1} (-1)^{n+1-i} b_{1i}^2 e_i$ for e_{n+2} in (5), we get

$$(5)' \quad \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, \dots, e_{n+1}] = e_2, \\ [e_2, \dots, \hat{e}_j, \dots, e_{n+2}] = b_{1j}^1 e_1, \quad 3 \leq j \leq n+1, \\ [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+2}] = b_{2j}^1 e_1 + b_{2j}^2 e_2, \quad 3 \leq j \leq n+1. \end{cases}$$

We discuss (5)' in two steps:

Step 1. If $b_{1j}^1 = 0$ for $3 \leq j \leq n+1$, then (5) is isomorphic to

$$(5)'' \quad \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{2j}^1 e_1 + b_{2j}^2 e_2, \quad (3 \leq j \leq n+1). \end{cases}$$

For every $j, 4 \leq j \leq n+1$, since

$b_{23}^1[e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+2}] = [[e_1, e_4, \dots, e_{n+2}], e_3, \dots, \hat{e}_j, \dots, e_{n+2}]$
 $= [b_{2j}^1 e_1 + b_{2j}^2 e_2, e_4, \dots, e_{n+2}] = b_{2j}^1 [e_1, e_4, \dots, e_{n+2}]$,
 we have $b_{23}^1 b_{2j}^2 = b_{23}^2 b_{2j}^1$. Then $[e_1, e_4, \dots, e_{n+2}]$ and $[e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+2}]$, $4 \leq j \leq n+1$ are linearly dependent.

If $[e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+2}] = 0$ for $3 \leq j \leq n+1$, then (5) is isomorphic to

$$(c^7) \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2. \end{cases}$$

If there exists $[e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+2}] \neq 0$ for some $3 \leq j \leq n+1$. We might as well suppose $[e_1, e_4, \dots, e_{n+2}] \neq 0$. Suppose $[e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+2}] = k_j [e_1, e_4, \dots, e_{n+2}]$, for $4 \leq j \leq n+1$. Substituting $e_3 - \sum_{j=4}^{n+1} (-1)^{i-4} k_j e_j$ for e_3 respectively in (5)'', we get

$$(5)''' \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, e_4, \dots, e_{n+1}, e_{n+2}] = b_{23}^1 e_1 + b_{23}^2 e_2, \end{cases} \quad \text{where } b_{23}^1 \neq 0 \text{ or } b_{23}^2 \neq 0. \text{ By suitable}$$

linear transformations for the basis e_1, \dots, e_{n+2} , we get (5)''' is isomorphic to (c^4) or (c^2) when $b_{23}^1 \neq 0$ or $b_{23}^2 \neq 0$ respectively.

Step 2. If there exists $b_{1j}^1 \neq 0$ for some $3 \leq j \leq n+1$. Then we might as well suppose $b_{13}^1 \neq 0$. Substituting $e_3 - \sum_{j=4}^{n+1} (-1)^{j-4} \frac{b_{1j}^1}{b_{13}^1} e_j$ and $\frac{1}{b_{13}^1} e_{n+2}$ for e_3 and e_{n+2} in (5)' respectively, we get (5) is isomorphic to

$$\begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2, \\ [e_2, e_4, \dots, e_{n+1}, e_{n+2}] = e_1, \\ [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{2j}^1 e_1 + b_{2j}^2 e_2, \quad 3 \leq j \leq n+1. \end{cases}$$

The discussions is completely similar to the Step 1, (5) is isomorphic to (c^2) , (c^4) or (c^6) for the cases that b_{2j}^1, b_{2j}^2 being to zero simultaneously or not.

Now we prove that (c^i) is not isomorphic to (c^j) when $i \neq j$ for $1 \leq i, j \leq 7$. The case (c^1) is not isomorphic to (c^3) , (c^5) and (c^7) since it is indecomposable. By Lemma 3.1 (c^i) is not isomorphic to (c^j) when $i \neq j$ for $i, j = 3, 5, 7$. And (c^j) for $j = 1, 3, 5, 7$ are not isomorphic to (c^2) , (c^4) , (c^6) since they have nonzero center.

For the cases (c^i) , $i = 2, 4, 6$, we have Lie algebras $A_i = A$ (as vector spaces) for $i = 2, 4, 6$ respectively with products $[\cdot, \cdot]_1$ as follows

$$(c^2)_1 \begin{cases} [e_2, e_3]_1 = e_1, \\ [e_2, e'_{n+2}]_1 = e_2, \\ [e_1, e'_{n+2}]_1 = e_1; \end{cases} \quad (c^4)_1 \begin{cases} [e_2, e_3]_1 = e_1, \\ [e_1, e_3]_1 = e_2, \\ [e_2, e'_{n+2}]_1 = e_2, \\ [e_1, e_{n+2}]_1 = e_1; \end{cases} \quad (c^6)_1 \begin{cases} [e_2, e_3]_1 = \alpha e_1 + e_2, \\ [e_1, e_3]_1 = e_2, \\ [e_2, e'_{n+2}]_1 = e_2, \\ [e_1, e'_{n+2}]_1 = e_1; \end{cases}$$

where $[x, y]_1 = [x, y, e_4, \dots, e_{n+1}]$ for $x, y \in A$ and $e'_{n+2} = (-1)^n e_{n+2}$. And A_i has decomposition $A_i = Z(A_i) \dot{+} B_i$ (the direct sum as ideals), where $B_i = Fe_1 + Fe_2 + Fe_3 + Fe_{n+2}$ for $i = 2, 4, 6$ are 4-dimensional solvable Lie algebras with multiplication table $(c^i)_1$ respectively.

It is easy to see that $H = Fe_3 + \dots + Fe_{n+2}$ is a Cartan subalgebra of (c^2) , (c^4) and (c^6) and the vectors e_4, \dots, e_{n+1} have the symmetric status in the multiplication. Then

(c^i) is isomorphic to (c^j) if and only if the Lie algebra $(c^i)_1$ is isomorphic to $(c^j)_1$. By the classification [23] of 4-dimensional solvable Lie algebras, $(c^i)_1$ is not isomorphic to $(c^j)_1$ for $i \neq j$. Then we get (c^i) is not isomorphic to (c^j) when $i \neq j$. And the n -Lie algebra of the case (c^6) with coefficient α is isomorphic to that with coefficient α' if and only if $\alpha = \alpha'$.

Summarizing, we get that (c^i) is not isomorphic to (c^j) if $i \neq j$ for $1 \leq i, j \leq 7$.

4. Let $\dim A^1 = 3$ and $A^1 = Fe_1 + Fe_2 + Fe_3$. By Lemma 3.1, Lemma 3.2 and Lemma 3.3, the multiplication table of A in a basis e_1, \dots, e_{n+2} has only following possibilities:

$$\begin{aligned}
(1) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2 + b_{ij}^3 e_3; \end{cases} \\
(2) \quad & \begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2 + b_{ij}^3 e_3; \end{cases} \\
(3) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2 + b_{ij}^3 e_3; \end{cases} \\
(4) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2 + b_{ij}^3 e_3; \end{cases} \\
(5) \quad & \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2 + b_{ij}^3 e_3; \end{cases} \\
(6) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, e_2, e_4, \dots, e_{n+1}] = e_3, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{ij}^1 e_1 + b_{ij}^2 e_2 + b_{ij}^3 e_3; \end{cases} \\
(7) \quad & \begin{cases} [e_3, e_4, \dots, e_{n+2}] = b_{12}^1 e_1 + b_{12}^2 e_2 + b_{12}^3 e_3, \\ [e_2, e_4, \dots, e_{n+2}] = b_{13}^1 e_1 + b_{13}^2 e_2 + b_{13}^3 e_3, \\ [e_1, e_4, \dots, e_{n+2}] = b_{23}^1 e_1 + b_{23}^2 e_2 + b_{23}^3 e_3, \end{cases}
\end{aligned}$$

where $b_{ij} \in F, 1 \leq i < j \leq n+1$.

Firstly, we study the case (1). Substituting the first identity into the other equations, we get

$$\begin{aligned}
& \sum_{k=1}^3 b_{ij}^k e_k = [e_1, e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\
& = [[e_2, \dots, e_{n+1}], e_2, e_3, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\
& = [e_2, e_3, \dots, (-1)^{i-2} \sum_{k=1}^3 b_{1j}^k e_k, \dots, e_{n+1}] + [e_2, e_3, \dots, (-1)^{j-3} \sum_{k=1}^3 b_{1i}^k e_k, \dots, e_{n+1}] \\
& = 0, \text{ for } 4 \leq i < j \leq n+1, \\
& \sum_{k=1}^3 b_{2j}^k e_k = [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}]
\end{aligned}$$

$$\begin{aligned}
&= [[e_2, \dots, e_{n+1}], e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{1j}^2 e_1, \text{ for } 4 \leq j \leq n+1, \\
&\sum_{k=1}^3 b_{3j}^k e_k = [e_1, e_2, e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\
&= [[e_2, \dots, e_{n+1}], e_2, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = -b_{1j}^3 e_1, \text{ for } 4 \leq j \leq n+1, \\
&\sum_{k=1}^3 b_{23}^k e_k = [e_1, e_4, \dots, e_{n+2}] \\
&= [[e_2, \dots, e_{n+1}], e_4, \dots, e_{n+2}] = b_{13}^2 e_1 + b_{12}^3 e_1.
\end{aligned}$$

If we replace $e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}^1 e_j$ for e_{n+2} , then (1) is isomorphic to

$$(1)' \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = \sum_{k=2}^3 b_{12}^k e_k, \\ [e_2, e_4, \dots, e_{n+2}] = \sum_{k=2}^3 b_{13}^k e_k, \\ [e_2, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=2}^3 b_{1j}^k e_k, \quad 4 \leq j \leq n+1, \\ [e_1, e_4, e_5, \dots, e_{n+2}] = b_{23}^1 e_1 = (b_{13}^2 + b_{12}^3) e_1, \\ [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{2j}^1 e_1 = b_{1j}^2 e_1, \quad 4 \leq j \leq n+1, \\ [e_1, e_2, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{3j}^1 e_1 = -b_{1j}^3 e_1, \quad 4 \leq j \leq n+1. \end{array} \right.$$

Fixing e_{n+2} in the n -ary multiplication of A , we get an $(n+2)$ -dimensional $(n-1)$ -Lie algebra $A_0 = A$ (as vector space) with production $[\dots]_0$ and the multiplication table of A_0 in the basis e_1, \dots, e_{n+2} is as follows

$$\left\{ \begin{array}{l} [e_3, e_4, \dots, e_{n+1}]_0 = \sum_{k=2}^3 b_{12}^k e_k, \\ [e_2, e_4, \dots, e_{n+1}]_0 = \sum_{k=2}^3 b_{13}^k e_k, \\ [e_2, e_3, \dots, \hat{e}_j, \dots, e_{n+1}]_0 = \sum_{k=2}^3 b_{1j}^k e_k, \quad 4 \leq j \leq n+1, \\ [e_1, e_4, e_5, \dots, e_{n+1}]_0 = b_{23}^1 e_1 = (b_{13}^2 + b_{12}^3) e_1, \\ [e_1, e_3, \dots, \hat{e}_j, \dots, e_{n+1}]_0 = b_{2j}^1 e_1 = b_{1j}^2 e_1, \quad 4 \leq j \leq n+1, \\ [e_1, e_2, \dots, \hat{e}_j, \dots, e_{n+1}]_0 = b_{3j}^1 e_1 = -b_{1j}^3 e_1, \quad 4 \leq j \leq n+1. \end{array} \right.$$

Set $B = Fe_2 + \dots + Fe_{n+1}$. Then B is a subalgebra of A_0 , $\dim B^1 = 2$ since $\dim A_0^1 = \dim A^1 = 3$, and the multiplication table of B in the basis e_2, \dots, e_{n+1} is as follows

$$\left\{ \begin{array}{l} [e_3, e_4, \dots, e_{n+1}]_0 = b_{12}^2 e_2 + b_{12}^3 e_3, \\ [e_2, e_4, \dots, e_{n+1}]_0 = b_{13}^2 e_2 + b_{13}^3 e_3, \\ [e_2, e_3, e_4, \dots, \hat{e}_j, \dots, e_{n+1}]_0 = b_{1j}^2 e_2 + b_{1j}^3 e_3, \quad 4 \leq j \leq n+1. \end{array} \right.$$

By discussions completely similar to [3], we have

$$\Delta = \begin{vmatrix} b_{12}^2 & b_{12}^3 \\ b_{13}^2 & b_{13}^3 \end{vmatrix} \neq 0, \text{ and } b_{1j}^2 = b_{1j}^3 = 0, \text{ for } 4 \leq j \leq n.$$

Therefore (1)' has the form

$$(1)'' \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = b_{12}^2 e_2 + b_{12}^3 e_3, \\ [e_2, e_4, \dots, e_{n+2}] = b_{13}^2 e_2 + b_{13}^3 e_3, \\ [e_1, e_4, e_5, \dots, e_{n+2}] = (b_{13}^2 + b_{12}^3) e_1, \end{array} \right. \quad \text{where } \Delta = \begin{vmatrix} b_{12}^2 & b_{12}^3 \\ b_{13}^2 & b_{13}^3 \end{vmatrix} \neq 0.$$

If $b_{13}^2 + b_{12}^3 = 0$, and $b_{12}^2 \neq 0$, taking a linear transformation for basis e_1, \dots, e_{n+2} by replacing $\frac{2\sqrt{\Delta}}{b_{12}^2} e_1$ for e_1 , $e_2 + \frac{b_{12}^3 - \sqrt{\Delta}}{b_{12}^2} e_3$ for e_2 , $e_2 + \frac{b_{12}^3 + \sqrt{\Delta}}{b_{12}^2} e_3$ for e_3 and $\frac{1}{\sqrt{\Delta}} e_{n+2}$ for e_{n+2} in (1)'', we get that (1) is isomorphic to

$$(d^1) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = -e_2, \\ [e_3, \dots, e_{n+2}] = e_3. \end{array} \right.$$

In the case that $b_{13}^2 + b_{12}^3 = b_{12}^2 = 0$, by discussions similar to above we get (1) is isomorphic to (d^1) .

If $b_{13}^2 + b_{12}^3 \neq 0$, and $b_{12}^2 \neq 0$, taking a linear transformation for basis e_1, \dots, e_{n+2} by replacing $\frac{\Delta}{b_{12}^2(b_{13}^2 + b_{12}^3)} e_1$ for e_1 , $e_2 + \frac{b_{12}^3}{b_{12}^2} e_3$ for e_2 , $e_2 + \frac{1}{b_{12}^2} (b_{12}^3 + \frac{\Delta}{b_{13}^2 + b_{12}^3}) e_3$ for e_3 , $\frac{1}{b_{13}^2 + b_{12}^3} e_{n+2}$ for e_{n+2} in (1)'', we get (1) isomorphic to

$$(d^2) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = e_3 + \alpha e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_1, e_4, \dots, e_{n+2}] = e_1, \end{array} \right. \quad \text{where } \alpha = \frac{\Delta}{(b_{13}^2 + b_{12}^3)^2} \in F \text{ and } \alpha \neq 0.$$

If $b_{13}^2 + b_{12}^3 \neq 0$ and $b_{12}^2 = 0$, (1)' is of the form

$$(1)''' \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = b_{12}^3 e_3, \\ [e_2, e_4, \dots, e_{n+2}] = b_{13}^2 e_2 + b_{13}^3 e_3, \\ [e_1, e_4, e_5, \dots, e_{n+2}] = (b_{13}^2 + b_{12}^3) e_1. \end{array} \right.$$

By similar discussions as above, taking suitable linear transformation for basis e_1, \dots, e_{n+2} , we get (1)''' is isomorphic to (d^2) in the case of $b_{13}^3 \neq 0$ or in the case of $b_{13}^3 = 0$ and $b_{13}^2 \neq b_{12}^3$. In the case of $b_{13}^3 = 0$ and $b_{13}^2 = b_{12}^3$, it is evident that (1)''' is of the form

$$(d^3) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, e_4, \dots, e_{n+2}] = e_3, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_1, e_4, \dots, e_{n+2}] = 2e_1. \end{array} \right.$$

Secondly substituting $e_{n+2} - \sum_{i=1}^n (-1)^{n-i} b_{in+1}^1 e_i$ for e_{n+2} in (2), we get

$$(2)' \left\{ \begin{array}{l} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = \sum_{k=1}^3 b_{ij}^k e_k, \quad 1 \leq i < j \leq n, \\ [e_1, \dots, \hat{e}_i, \dots, e_n, e_{n+2}] = \sum_{k=2}^3 b_{in+1}^k e_k, \quad 1 \leq i \leq n. \end{array} \right.$$

Using the Jacobi identities for $\{[e_1, \dots, e_n], e_3, \dots, e_n, e_{n+2}\}$, $\{[e_1, \dots, e_n], e_2, e_4, \dots,$

$e_n, e_{n+2}\}$, $\{[e_1, \dots, e_n], e_2, e_3, e_4, \dots, \hat{e}_i, \dots, e_n, e_{n+2}\}$ for $4 \leq i \leq n$, $\{[e_1, e_2, e_4, \dots, e_n, e_{n+2}], e_2, \dots, \hat{e}_j, \dots, e_{n+1}\}$ for $2 \leq j \leq n$, $\{[e_2, \dots, e_{n+1}], e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}\}$ for $2 \leq i \leq j \leq n$, we get $b_{ij}^3 = 0$ for $1 \leq i < j \leq n$. Hence (2) is of the form

$$\begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = \sum_{k=1}^2 b_{ij}^k e_k, \quad 1 \leq i < j \leq n. \end{cases}$$

This contradicts $\dim A^1 = 3$. Therefore the case (2) is not realized.

Thirdly, imposing the Jacobi identities on (3) for $\{[e_2, \dots, e_{n+1}], e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}\}$ for $2 \leq i < j \leq n+1$, $\{[e_1, e_3, \dots, e_{n+1}], e_3, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$ for $3 \leq j \leq n+1$, $\{[e_1, e_3, \dots, e_{n+1}], e_4, \dots, e_{n+2}\}$ and $\{[e_2, \dots, e_{n+1}], e_4, \dots, e_{n+2}\}$, we get $b_{ij}^3 = 0$ for $2 \leq i < j \leq n+1$, $b_{1j}^3 = 0$ for $3 \leq j \leq n+1$, $b_{12}^3 = 0$ respectively. Then (3) is of the form

$$\begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_n] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^2 b_{ij}^k e_k, \quad 1 \leq i < j \leq n+1. \end{cases}$$

This contradicts $\dim A^1 = 3$. Therefore, the case (3) is not realized.

The cases (4) and (5) are not realized by discussions similar to the case (3).

Fourthly, for $4 \leq i < j \leq n+1$, from the table (6)

$$\begin{aligned} \sum_{k=1}^3 b_{ij}^k e_k &= [e_1, e_2, e_3, e_4, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\ &= [[e_2, \dots, e_{n+1}], e_2, e_3, e_4, e_5, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] \\ &= [e_2, e_3, e_4, \dots, (-1)^{i-2} \sum_{k=1}^3 b_{1j}^k e_k, \dots, e_{n+1}] \\ &+ [e_2, e_3, e_4, \dots, (-1)^{j-3} \sum_{k=1}^3 b_{1i}^k e_k, \dots, e_{n+1}] = 0. \end{aligned}$$

Then (6) has the form

$$(6)' \quad \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_n] = e_2, \\ [e_1, e_2, e_4, \dots, e_{n+1}] = e_3, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^3 b_{ij}^k e_k, \quad 1 \leq i \leq 3, i < j \leq n+1. \end{cases}$$

For $4 \leq j \leq n+1$, imposing the Jacobi identities for $\{[e_1, e_3, \dots, e_{n+1}], e_3, e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$, $\{e_2, [e_1, e_2, e_4, \dots, e_{n+1}], e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$, $\{[e_2, \dots, e_{n+1}], e_3, e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$, $\{e_1, [e_1, e_2, e_4, \dots, e_{n+1}], e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$, $\{[e_2, \dots, e_{n+1}], e_2, e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$, $\{e_1, [e_1, e_3, \dots, e_{n+1}], e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$, $\{[e_1, e_2, e_4, \dots, e_{n+1}], e_4, \dots, e_{n+2}\}$, $\{[e_1, e_3, \dots, e_{n+1}], e_4, \dots, e_{n+2}\}$, $\{[e_2, \dots, e_{n+1}], e_4, \dots, e_{n+2}\}$, we get $b_{1j}^3 = 0$, $b_{1j}^1 = b_{2j}^2$, $b_{1j}^2 = b_{2j}^1$; $b_{1j}^2 = 0$, $b_{1j}^3 = -b_{3j}^1$, $b_{1j}^1 = b_{3j}^3$; $b_{2j}^3 = 0$, $b_{2j}^1 = b_{1j}^2$, $b_{2j}^2 = b_{1j}^1$; $b_{2j}^1 = 0$, $b_{2j}^2 = b_{3j}^3$, $b_{2j}^3 = b_{3j}^2$; $b_{3j}^2 = 0$, $b_{3j}^1 = -b_{1j}^3$,

$b_{3j}^3 = b_{1j}^1$; $b_{3j}^1 = 0$, $b_{3j}^2 = b_{2j}^3$, $b_{3j}^3 = b_{2j}^2$; $b_{12}^1 = -b_{23}^3$, $b_{12}^2 = b_{13}^3$, $b_{12}^3 = b_{13}^2 + b_{23}^1$; $b_{13}^1 = b_{23}^2$, $b_{13}^2 = b_{12}^3 + b_{23}^1$, $b_{13}^3 = b_{12}^2$; $b_{23}^1 = b_{12}^3 + b_{13}^2$, $b_{23}^2 = b_{13}^1$, $b_{23}^3 = -b_{12}^1$ respectively. Then (6)' is of the form

$$\left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, e_2, e_4, \dots, e_{n+1}] = e_3, \\ [e_3, \dots, e_{n+2}] = b_{12}^1 e_1 + b_{12}^2 e_2, \\ [e_2, e_4, \dots, e_{n+2}] = b_{13}^1 e_1 + b_{12}^2 e_3, \\ [e_1, e_4, e_5, \dots, e_{n+2}] = b_{13}^1 e_2 - b_{12}^1 e_3, \\ [e_2, e_3, e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{1j}^1 e_1, \quad 4 \leq j \leq n+1, \\ [e_1, e_3, e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{2j}^2 e_2 = b_{1j}^1 e_2, \quad 4 \leq j \leq n+1, \\ [e_1, e_2, e_4, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{3j}^3 e_3 = b_{1j}^1 e_3, \quad 4 \leq j \leq n+1. \end{array} \right.$$

Substituting $e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}^1 e_j - (-1)^{n-1} b_{12}^2 e_1$ for e_{n+2} , we get (6) is isomorphic to

$$(d^4) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, e_2, e_4, \dots, e_{n+1}] = e_3. \end{array} \right.$$

Lastly we discuss the case (7). It follows by a simple computation that there does not exist any nonabelian proper subalgebra of A containing A^1 . Then the multiplication of A is completely determined by the left multiplication $\text{ad}(e_4, \dots, e_{n+2})$. And $\text{ad}(e_4, \dots, e_{n+2})|_{A^1}$ is nonsingular since $\dim A^1 = 3$. So we can choose a basis e_1, e_2, e_3 of A^1 such that the multiplication of A in the basis e_1, \dots, e_{n+2} has the following possibilities

$$\begin{aligned} (d^5)' & \left\{ \begin{array}{l} [e_1, e_4, \dots, e_{n+2}] = \beta_1 e_1, \\ [e_2, e_4, \dots, e_{n+2}] = \beta_2 e_2, \\ [e_3, e_4, \dots, e_{n+2}] = \beta_3 e_3, \end{array} \quad \beta_i \in F, \beta_i \neq 0, i = 1, 2, 3; \right. \\ (d^6)' & \left\{ \begin{array}{l} [e_1, e_4, \dots, e_{n+2}] = \alpha e_1 + e_2, \\ [e_2, e_4, \dots, e_{n+2}] = \alpha e_2 + e_3, \\ [e_3, e_4, \dots, e_{n+2}] = \alpha e_3, \end{array} \quad \alpha \in F, \alpha \neq 0; \right. \\ (d^7)' & \left\{ \begin{array}{l} [e_1, e_4, \dots, e_{n+2}] = \gamma_1 e_1 + e_2, \\ [e_2, e_4, \dots, e_{n+2}] = \gamma_1 e_2, \\ [e_3, e_4, \dots, e_{n+2}] = \gamma_2 e_3, \end{array} \quad \gamma_j \in F, \gamma_j \neq 0, j = 1, 2. \right. \end{aligned}$$

If we fix e_5, \dots, e_{n+2} in the n -ary multiplication of A , we get solvable Lie algebra $A_1 = A$ (as vector spaces) with the Lie production $[\cdot]_1$

$$[x, y]_1 = [x, y, e_5, \dots, e_{n+2}], \quad x, y \in A_1.$$

Then the multiplication tables of A_1 with respect to $(d^5)', (d^6)', (d^7)'$ are

$$(d^5)'' \left\{ \begin{array}{l} [e_1, e_4]_1 = \beta_1 e_1, \\ [e_2, e_4]_1 = \beta_2 e_2, \\ [e_3, e_4]_1 = \beta_3 e_3, \end{array} \quad \beta_i \in F, \beta_i \neq 0, i = 1, 2, 3; \right.$$

$$(d^6)'' \begin{cases} [e_1, e_4]_1 = \alpha e_1 + e_2, \\ [e_2, e_4]_1 = \alpha e_2 + e_3, \\ [e_3, e_4]_1 = \alpha e_3, \end{cases} \quad \alpha \in F, \alpha \neq 0;$$

$$(d^7)'' \begin{cases} [e_1, e_4]_1 = \gamma_1 e_1 + e_2, \\ [e_2, e_4]_1 = \gamma_1 e_2, \\ [e_3, e_4]_1 = \gamma_2 e_3, \end{cases} \quad \gamma_j \in F, \gamma_j \neq 0, j = 1, 2.$$

This implies that $(d^i)''$ can be decomposed into the direct sum of ideals $Z(A_1)$ and B , where the center $Z(A_1) = Fe_5 + \cdots + Fe_{n+2}$ and the ideal $B = Fe_1 + Fe_2 + Fe_3 + Fe_4$, $i = 5, 6, 7$. By the classification of 4-dimensional solvable Lie algebras [23], we get that one and only one of following possibilities holds up to isomorphisms:

$$(d^5) \begin{cases} [e_1, e_4, \dots, e_{n+2}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_3, e_4, \dots, e_{n+2}] = \beta e_2 + (1 + \beta)e_3, \end{cases} \quad \beta \in F, \beta \neq 0, 1;$$

$$(d^6) \begin{cases} [e_1, e_4, \dots, e_{n+2}] = e_1, \\ [e_2, e_4, \dots, e_{n+2}] = e_2, \\ [e_3, e_4, \dots, e_{n+2}] = e_3; \end{cases}$$

$$(d^7) \begin{cases} [e_1, e_4, \dots, e_{n+2}] = e_2, \\ [e_2, e_4, \dots, e_{n+2}] = e_3, \\ [e_3, e_4, \dots, e_{n+2}] = se_1 + te_2 + ue_3, \end{cases} \quad s, t, u \in F, s \neq 0.$$

And (d^i) is not isomorphic to (d^j) when $i \neq j$ for $5 \leq i, j \leq 7$. And the n -Lie algebras corresponding to the case (d^5) with coefficients β and β' are isomorphic if and only if $\beta = \beta'$. We also have that the n -Lie algebras corresponding to the case (d^7) with coefficients s, t, u and s', t', u' are isomorphic if and only if there exists a nonzero element $r \in F$ such that

$$s = r^3 s', \quad t = r^2 t', \quad u = ru', \quad s, s', t, t', u, u' \in F.$$

It is evident that $(d^5), (d^6), (d^7)$ are not isomorphic to other cases since $(d^5), (d^6), (d^7)$ have no nonabelian proper subalgebras containing A^1 . The case (d^4) is not isomorphic to any cases of $(d^1), (d^2), (d^3)$ since (d^4) is decomposable. Because (d^1) has non-trivial center, (d^1) is not isomorphic to (d^2) and (d^3) . By direct computation we know that dimensions of the derivation algebras of the case (d^2) and (d^3) are $n^2 + 2$ and $n^2 + 3$ respectively, therefore, (d^2) and d^3 represent non-isomorphic classes.

Now fixing e_4, \dots, e_{n+1} in the multiplication of A of the case (d^2) , and substituting $(-1)^{n-2}e_{n+2}$ for e_{n+2} , we get a solvable Lie algebra A_2 ($A_2 = A$ as vector spaces) with the product

$$[x, y]_2 = [x, y, e_4, \dots, e_{n+1}],$$

and the multiplication table of A_2 in the basis e_1, \dots, e_{n+2} is as follows

$$\begin{cases} [e_2, e_3]_2 = e_1, \\ [e_3, e_{n+2}]_2 = e_3 + \alpha e_2, \\ [e_2, e_{n+2}]_2 = e_2, \\ [e_1, e_{n+2}]_2 = e_1. \end{cases}$$

Then A_2 has a decomposition $A_2 = B \oplus Z(A_2)$, where the center $Z(A_2) = Fe_1 + \cdots + Fe_{n+1}$, and $B = Fe_1 + Fe_2 + Fe_3 + Fe_{n+2}$ is an ideal. By the classification of solvable Lie algebras [23], we get that n -Lie algebras of the case (d^2) with coefficients α and α' are isomorphic if and only if $\alpha = \alpha'$.

5. By Lemma 3.3, we have $\dim A^1 = r$, $4 \leq r \leq n+1$. Suppose $A^1 = Fe_1 + \cdots + Fe_r$. From Lemma 3.1 and Lemma 3.2, the multiplication table of A in the basis e_1, \dots, e_{n+2} has following possibilities

$$\begin{aligned}
(1) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k; \end{cases} \\
(2) \quad & \begin{cases} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k; \end{cases} \\
(3) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = e_1, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k; \end{cases} \\
(4) \quad & \begin{cases} [e_2, \dots, e_{n+1}] = \alpha e_1 + e_2, \\ [e_1, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k; \end{cases} \\
(5) \quad & \begin{cases} [e_1, e_3, \dots, e_{n+1}] = e_1, \\ [e_2, e_3, \dots, e_{n+1}] = e_2, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k; \end{cases} \\
(6) \quad & \begin{cases} [e_1, \dots, \hat{e}_i, \dots, e_m, \dots, e_r, \dots, e_{n+1}] = e_i, \quad 1 \leq i \leq m, 3 \leq m \leq r-1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k; \end{cases} \\
(7) \quad & \begin{cases} [e_1, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, e_{n+1}] = e_i, \quad 1 \leq i \leq r, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k; \end{cases}
\end{aligned}$$

where $b_{ij} \in F$, $1 \leq i < j \leq n+1$.

Firstly, we study the case (1). For substituting $[e_2, \dots, e_{n+1}] = e_1$ into the other equations and by the Jacobi identities on $\{[e_2, \dots, e_{n+1}], e_2, \dots, e_r, e_{r+1}, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$ for $r+1 \leq i < j \leq n+1$; $\{[e_2, \dots, e_{n+1}], e_2, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$ for $2 \leq i \leq r$, $r+1 \leq j \leq n+1$; $\{[e_2, \dots, e_{n+1}], e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_r, e_{r+1}, \dots, e_{n+2}\}$ for $2 \leq i < j \leq r$, we get $b_{ij}^k = 0$ for $1 \leq k \leq r$, $r+1 \leq i < j \leq n+1$; $b_{ij}^1 = (-1)^{i-2} b_{1j}^i$, $b_{ij}^2 = \dots = b_{ij}^r = 0$ for $2 \leq i \leq r$, $r+1 \leq j \leq n+1$; and $b_{ij}^1 = (-1)^{i-2} b_{1j}^i + (-1)^{j-3} b_{1i}^j$, $b_{ij}^2 = \dots = b_{ij}^r = 0$ for $2 \leq i < j \leq r$ respectively.

Replacing $e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}^1 e_j$ for e_{n+2} , we get the isomorphic form of (1)

$$(1)' \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_2, \dots, \hat{e}_j, \dots, e_r, \dots, e_{n+2}] = \sum_{k=2}^r b_{1j}^k e_k, 2 \leq j \leq r, \\ [e_2, \dots, e_r, e_{r+1}, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=2}^r b_{1j}^k e_k, r+1 \leq j \leq n+1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_r, \dots, e_{n+2}] = ((-1)^{i-2} b_{1j}^i + (-1)^{j-3} b_{1i}^j) e_1, 2 \leq i < j \leq r, \\ [e_1, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, \hat{e}_j, \dots, e_{n+2}] = (-1)^i b_{1j}^i e_1, 2 \leq i \leq r < j \leq n+1. \end{array} \right.$$

If fixing e_{n+2} in the multiplication of A , we get an $(n+2)$ -dimensional $(n-1)$ -Lie algebra $A_3 = A$ (as vector spaces) with the product $[x_1, \dots, x_{n-1}]_3 = [x_1, \dots, x_{n-1}, e_{n+2}]$ for $\forall x_1, \dots, x_{n-1} \in A_3$, and the multiplication table in the basis e_1, \dots, e_{n+2} is as follows

$$\left\{ \begin{array}{l} [e_2, \dots, \hat{e}_j, \dots, e_r, \dots, e_{n+1}]_3 = \sum_{k=2}^r b_{1j}^k e_k, 2 \leq j \leq r, \\ [e_2, \dots, e_r, e_{r+1}, \dots, \hat{e}_j, \dots, e_{n+1}]_3 = \sum_{k=2}^r b_{1j}^k e_k, r+1 \leq j \leq n+1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_r, \dots, e_{n+1}]_3 = ((-1)^i b_{1j}^i + (-1)^{j-1} b_{1i}^j) e_1, 2 \leq i < j \leq r, \\ [e_1, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, \hat{e}_j, \dots, e_{n+1}]_3 = (-1)^i b_{1j}^i e_1, 2 \leq i \leq r < j \leq n+1. \end{array} \right.$$

Set $B = Fe_2 + \dots + Fe_{n+1}$. Then B is a subalgebra of A_3 with multiplication table

$$(1_b) \left\{ \begin{array}{l} [e_2, \dots, \hat{e}_j, \dots, e_r, \dots, e_{n+1}]_3 = \sum_{k=2}^r b_{1j}^k e_k, 2 \leq j \leq r, \\ [e_2, \dots, e_r, e_{r+1}, \dots, \hat{e}_j, \dots, e_{n+1}]_3 = \sum_{k=2}^r b_{1j}^k e_k, r+1 \leq j \leq n+1. \end{array} \right.$$

By similar discussions leading to Theorem 3 in [3], we get

$$b_{1j}^2 = \dots = b_{1j}^r = 0, r+1 \leq j \leq n+1; (-1)^{i-2} b_{1j}^i + (-1)^{j-3} b_{1i}^j = 0.$$

Taking a suitable transformation of basis e_2, \dots, e_{n+1} , we get (1_b) isomorphic to

$$\left\{ \begin{array}{l} [e_3, \dots, e_{n+1}]_3 = e_2, \\ \dots \quad \dots \quad \dots \\ [e_2, \dots, \hat{e}_i, \dots, e_{n+1}]_3 = e_i, 2 \leq i \leq r, \\ \dots \quad \dots \quad \dots \\ [e_3, \dots, e_{r-1}, e_{r+1}, \dots, e_{n+1}]_3 = e_r. \end{array} \right.$$

Therefore, after taking a suitable e_1 , we get (1) is isomorphic to

$$(r^1) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ [e_3, \dots, e_{n+2}] = e_2, \\ \dots \quad \dots \quad \dots \quad \dots, \\ [e_2, \dots, \hat{e}_i, \dots, e_r, \dots, e_{n+2}] = e_i, \\ \dots \quad \dots \quad \dots \quad \dots, \\ [e_2, \dots, e_{r-1}, e_{r+1}, \dots, e_{n+2}] = e_r. \end{array} \right.$$

Secondly, we study the case (2). Replacing e_{n+2} by $e_{n+2} - \sum_{i=1}^n (-1)^{n-i} b_{in+1}^1 e_i$ in (2), we get

$$(2)' \left\{ \begin{array}{l} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_n, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k, \quad 1 \leq i < j \leq n, \\ [e_1, \dots, \hat{e}_i, \dots, e_n, e_{n+2}] = \sum_{k=2}^r b_{in+1}^k e_k, \quad 1 \leq i \leq n. \end{array} \right.$$

Substituting $[e_1, \dots, e_n] = e_1$ into other equations and by the Jacobi identities on $\{[e_1, \dots, e_n], e_2, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, e_n, e_{n+2}\}$ for $2 \leq i \leq r$, $\{[e_1, \dots, e_n], e_2, \dots, e_r, e_{r+1}, \dots, \hat{e}_i, \dots, e_n, e_{n+2}\}$ for $r+1 \leq i \leq n$, $\{[e_1, \dots, e_{r-1}, e_{r+1}, \dots, e_n, e_{n+2}], e_2, \dots, \hat{e}_j, \dots, e_{n+1}\}$ for $i=1, 2 \leq j \leq n$, $\{[e_1, \dots, e_n], e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$ for $2 \leq i < j \leq n$, we get $b_{in+1}^2 = \dots = b_{in+1}^r = 0, b_{1n+1}^i = 0$ for $2 \leq i \leq r$, $b_{in+1}^2 = \dots = b_{in+1}^r = 0$ for $r+1 \leq i \leq n$, $b_{1j}^r = 0$ for $i=1, 2 \leq j \leq n$, $b_{ij}^r = 0$ for $2 \leq i < j \leq n$ respectively. Then (2) is of the form

$$\left\{ \begin{array}{l} [e_1, \dots, e_n] = e_1, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^{r-1} b_{ij}^k e_k, \quad 1 \leq i < j \leq n. \end{array} \right.$$

We get $\dim A^1 = r-1$, this is a contradiction. Therefore the case (2) is not realized.

By similar arguments to above, we get that cases (3), (4), (5) and (6) are not realized.

Lastly we study the case (7). For $r+1 \leq i < j \leq n+1$, imposing the Jacobi identities on $\{[e_2, \dots, e_{n+1}], e_2, \dots, e_r, e_{r+1}, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}\}$, we get $b_{ij}^k = 0, 1 \leq k \leq r$. Then (7) has the form

$$\left\{ \begin{array}{l} [e_1, \dots, \hat{e}_i, \dots, e_r, \dots, e_{n+1}] = e_i, \quad 1 \leq i \leq r, \\ [e_1, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = \sum_{k=1}^r b_{ij}^k e_k, \quad 1 \leq i \leq r, i < j \leq n+1. \end{array} \right.$$

For every $i \neq p, 1 \leq i, p \leq r$, substituting $[e_1, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, e_{n+1}] = e_i$ into the equation

$$\sum_{k=1}^r b_{ij}^k e_k = [e_1, \dots, e_{p-1}, e_{p+1}, \dots, e_r, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}], \quad r+1 \leq j \leq n+1, \text{ we get}$$

$$b_{pj}^p = b_{1j}^1, \quad b_{pj}^k = 0 \quad \text{for } k \neq p, \quad 1 \leq p, \quad k \leq r < j \leq n+1.$$

For $2 \leq j \leq r$; substituting $[e_1, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, e_{n+1}] = e_i, 2 \leq i \neq j \leq r$ into the equation

$$\sum_{k=1}^r b_{1j}^k e_k = [e_2, \dots, \hat{e}_j, \dots, e_r, e_{r+1}, \dots, e_{n+2}],$$

we get

$$b_{1j}^k = 0, \quad b_{1j}^1 = \dots = b_{j-1j}^{j-1} = -b_{jj+1}^{j+1} = \dots = -b_{jr}^r, \quad b_{1j}^j = b_{12}^2 \quad \text{for } 2 \leq k \neq j \leq r.$$

For $2 \leq i < j \leq r$, substituting $[e_1, \dots, \hat{e}_p, \dots, e_{n+1}] = e_p, 1 \leq p \leq r$ and $p \neq i, j$ into the equation

$$\sum_{k=1}^r b_{ij}^k e_k = [e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_r, \dots, e_{n+2}],$$

we get

$$b_{ij}^k = 0 \quad \text{for } 1 \leq k \leq r, \quad k \neq i, j, \quad \text{and } 2 \leq i < j \leq r.$$

Then (7) is isomorphic to

$$(7)' \left\{ \begin{array}{l} [e_1, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, e_{n+1}] = e_i, \quad 1 \leq i \leq r, \\ [e_2, \dots, \hat{e}_j, \dots, e_r, e_{r+1}, \dots, e_{n+2}] = b_{1j}^1 e_1 + b_{12}^2 e_j, \quad 2 \leq j \leq r, \\ [e_1, e_2, \dots, \hat{e}_i, \dots, \hat{e}_j, \dots, e_r, e_{r+1}, \dots, e_{n+2}] = b_{1j}^1 e_i - b_{1i}^1 e_j, \quad 2 \leq i < j \leq r, \\ [e_1, \dots, \hat{e}_i, \dots, e_r, e_{r+1}, \dots, \hat{e}_j, \dots, e_{n+1}, e_{n+2}] = b_{1j}^1 e_i, \quad 1 \leq i \leq r < j \leq n+1. \end{array} \right.$$

Replacing e_{n+2} by $e_{n+2} - \sum_{j=2}^{n+1} (-1)^{n+1-j} b_{1j}^1 e_j - (-1)^{n-1} b_{12}^2 e_1$ in (7)', we get

$$(r^2) \left\{ \begin{array}{l} [e_2, \dots, e_{n+1}] = e_1, \\ \dots \quad \dots \quad \dots \quad \dots, \\ [e_1, \dots, \hat{e}_i, \dots, e_r, \dots, e_{n+1}] = e_i, \\ \dots \quad \dots \quad \dots \quad \dots, \\ [e_1, \dots, e_{r-1}, e_{r+1}, \dots, e_{n+1}] = e_r. \end{array} \right.$$

It is evident that (r^1) is not isomorphic to (r^2) . □

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